This week

GELGOOG Coffee Beans Husking Machine

1. Section 1.1: functions and their graphs
2. Section 1.2: combining and transforming functions
3. Section 1.3: trigonometry
Colstructie = College + Instructie
About this course

- Colstructie = College + Instructie = “Lectorial”.

Three midterm tests and one resit. See MyTimeTable for date and time. Tests and exercises with MyLabsPlus. Examples with Mathematica. Course schedule, slides and other materials can be found on Blackboard page 2017-201700041-1B: Smart Environments (2017-1B), item Course Materials, folder ItE: Introduction to Mathematics and Modeling I.
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<table>
<thead>
<tr>
<th>Nr</th>
<th>Week</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Basics: functions, graphs and trigonometry</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>Basics: the inverse; exponential functions and logarithms</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Midterm test 1</strong></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>Differentiation: definition</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>Differentiation: rules and properties</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>Differentiation: applications</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Midterm test 2</strong></td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>Integration: definition and applications</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>Integration: the fundamental theorem; method of substitution</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>Integration: integration by parts</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Midterm test 3</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Resit</strong></td>
</tr>
</tbody>
</table>
Definition

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- If $f$ assigns $y$ to $x$, then we denote this as $y = f(x)$ or $x \mapsto f(x)$. 
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- If \( f \) assigns \( y \) to \( x \), then we denote this as \( y = f(x) \) or \( x \mapsto f(x) \).
- The object \( f(x) \) is called the **image of** \( x \) (**under** \( f \)).
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- The object $f(x)$ is called the **image of** $x$ (under $f$).
- Synonyms for ‘function’ are **map** or **transformation**.
- Sometimes we use a **diagram**:

  $x \overset{f}{\longrightarrow} f(x)$
Definition

Let $D$ and $Y$ are subsets of $\mathbb{R}$. The graph of a function $f: D \to Y$ is defined as

$$\text{graph}(f) = \{(x, f(x)) \mid x \in D\}.$$
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The graph of a function

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Vertical Line Test

A vertical line intersects the graph of a function in at most one point.
### Mathematica

- **Defining a function:**

  
  \[ f[x_] := \frac{1}{\sqrt{x^2 + 1}} \]

- **Plotting a function \( f \) with domain \([a, b]\):**

  
  \[
  \text{Plot}[f[x], \{x, a, b\}] 
  \]
Equality of functions

Definition

Two functions \( f \) and \( g \) are equal if

1. the domain of \( f \) is equal to the domain of \( g \), and
2. if \( f(x) = g(x) \) for all \( x \) in the domain of \( f \).
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- Sometimes the equality of the codomains is also required.
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2. if $f(x) = g(x)$ for all $x$ in the domain of $f$.

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Definition

Let the function $f$ be defined by a formula.

- If the domain of $f$ is not defined explicitly, then the domain consists of all numbers $x$ for which $f(x)$ exists.
- If the codomain of $f$ is not defined explicitly, then the codomain is chosen as large as possible.
Definition

Let the function $f$ be defined by a formula.

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- If the codomain of $f$ is not defined explicitly, then the codomain is chosen as large as possible.

Example:

Let $f(x) = \sqrt{x - 3}$.

- The expression $\sqrt{x - 3}$ is defined for all $x$ for which $x - 3 \geq 0$, hence $\text{Dom}(f) = [3, \infty)$.
- The codomain is $\mathbb{R}$.
If a domain consists of several parts $D_1, D_2, \ldots, D_n$, a function may be defined with different formulas per part:

$$f(x) = \begin{cases} 
F_1(x) & \text{if } x \in D_1, \\
F_2(x) & \text{if } x \in D_2, \\
\vdots & \vdots \\
F_n(x) & \text{if } x \in D_n. 
\end{cases}$$

**Example**

The absolute value is defined as

$$|x| = \begin{cases} 
x & \text{if } x \geq 0, \\
-x & \text{if } x < 0.
\end{cases}$$
Monotony

**Definition**

Let $f : I \to \mathbb{R}$ be a function defined on an interval $I$.

1. **The function $f$ is increasing** if for all $x_1, x_2 \in I$ with $x_1 < x_2$ we have $f(x_1) < f(x_2)$.

2. **The function $f$ is decreasing** if for all $x_1, x_2 \in I$ with $x_1 < x_2$ we have $f(x_1) > f(x_2)$.
Monotony

**Definition**

Let \( f : I \to \mathbb{R} \) be a function defined on an interval \( I \).

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2. The function \( f \) is **decreasing** if for all \( x_1, x_2 \in I \) with \( x_1 < x_2 \) we have \( f(x_1) > f(x_2) \).

- A function that is decreasing or increasing is called **monotonous**.
**Definition**

A subset $D \subset \mathbb{R}$ is **symmetric** if for all $x \in D$ also $-x \in D$. 

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**Example**

$[-1, 1]$ and $\mathbb{R}\{0\}$ are symmetric.

$[0, 1]$ and $[-1, 1)$ are not symmetric.
Definition

A subset $D \subset \mathbb{R}$ is symmetric if for all $x \in D$ also $-x \in D$.

Example

- $\mathbb{R}$, $[-1, 1]$ and $\mathbb{R}\{0\}$ are symmetric.
Definition

A subset \( D \subset \mathbb{R} \) is **symmetric** if for all \( x \in D \) also \( -x \in D \).

Example

- \( \mathbb{R} \), \([-1, 1]\) and \( \mathbb{R} \setminus \{0\} \) are symmetric.
- \([0, 1]\) en \([-1, 1)\) are not symmetric.
**Definition**

A subset $D \subset \mathbb{R}$ is **symmetric** if for all $x \in D$ also $-x \in D$.

**Example**

- $\mathbb{R}$, $[-1, 1]$ and $\mathbb{R}\{0\}$ are symmetric.
- $[0, 1]$ en $[-1, 1)$ are not symmetric.

**Definition**

Let $D$ be a symmetric subset of $\mathbb{R}$.

- A function $f : D \rightarrow \mathbb{R}$ is **even** if $f(-x) = f(x)$ for all $x \in D$.
- A function $f : D \rightarrow \mathbb{R}$ is **odd** if $f(-x) = -f(x)$ for all $x \in D$. 
The graph of an even function is symmetric about the vertical axis.

The graph of an odd function is symmetric about the origin.
The graph of an even function is symmetric about the vertical axis.

\[ f(-x) = f(x) \]
The graph of an even function is symmetric about the vertical axis.

\[ f(-x) = f(x) \]

The graph of an odd function is symmetric about the origin.

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\[ f(-x) = -f(x) \]
Assignment: IMM1 - Tutorial 1.1
Algebraic combinations

2.1

Addition: \[ h(x) = f(x) + g(x) \quad f + g \]
Algebraic combinations

2.1

Addition: \( h(x) = f(x) + g(x) \)  \( f + g \)

Subtraction: \( h(x) = f(x) - g(x) \)  \( f - g \)
Algebraic combinations

Addition: \[ h(x) = f(x) + g(x) \quad f + g \]

Subtraction: \[ h(x) = f(x) - g(x) \quad f - g \]

Multiplication: \[ h(x) = f(x)g(x) \quad fg \]
Algebraic combinations

2.1

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Subtraction: \[ h(x) = f(x) - g(x) \quad f - g \]

Multiplication: \[ h(x) = f(x)g(x) \quad fg \]

Division: \[ h(x) = \frac{f(x)}{g(x)} \quad \frac{f}{g} \]
Algebraic combinations

Addition: \[ h(x) = f(x) + g(x) \] 
\[ f + g \]

Subtraction: \[ h(x) = f(x) - g(x) \] 
\[ f - g \]

Multiplication: \[ h(x) = f(x)g(x) \] 
\[ fg \]

Division: \[ h(x) = \frac{f(x)}{g(x)} \] 
\[ \frac{f}{g} \]

Composition: \[ h(x) = f(g(x)) \] 
\[ f \circ g \]
Let \( f: D \rightarrow C \) and \( g: E \rightarrow D \) be two functions, where the domain of \( f \) is the codomain of \( g \).
Let $f : D \to C$ and $g : E \to D$ be two functions, where the domain of $f$ is the codomain of $g$. 

**Diagram:** 
- $x \in E$ 
- $g(x) \in D$ 
- $f(g(x)) \in C$ 

The composition of $f$ and $g$ is defined as the function $f \circ g : E \to C$ that assigns the element $f(g(x))$ to every $x \in E$. Pronounce $f \circ g$ as "$f$ after $g". 

**Arrow diagram:** 
- $x \rightarrow g(x) \rightarrow f(g(x))$
Let $f : D \to C$ and $g : E \to D$ be two functions, where the domain of $f$ is the codomain of $g$. 

**Diagram:**

- $x \in E$ 
- $g(x) \in D$ 
- $f(g(x)) \in C$ 

The composition $f \circ g : E \to C$ is defined as the function that assigns the element $f(g(x))$ to every $x \in E$. 

Pronounce $f \circ g$ as "$f$ after $g$".
Let $f: D \rightarrow C$ and $g: E \rightarrow D$ be two functions, where the domain of $f$ is the codomain of $g$.

The **composition** of $f$ and $g$ is defined as the function $f \circ g: E \rightarrow C$ that assigns the element $f(g(x))$ to every $x \in E$. 

\begin{align*}
f \circ g : E &\longrightarrow C \\
g : E &\rightarrow D \\
f : D &\rightarrow C \\
\end{align*}
Composition

Let \( f : D \to C \) and \( g : E \to D \) be two functions, where the domain of \( f \) is the codomain of \( g \).

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Pronounce \( f \circ g \) as “\( f \) after \( g \)”. 

\[
\begin{array}{c}
\text{Let } f : D \to C \text{ and } g : E \to D \text{ be two functions, where the domain of } f \text{ is the codomain of } g. \\
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\end{array}
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Arrow diagram:
Let $f: D \to C$ and $g: E \to D$ be two functions, where the domain of $f$ is the codomain of $g$.

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Pronounce $f \circ g$ as “$f$ after $g$”.

Arrow diagram:
Example

Define \( f(x) = \sqrt{x} \) and \( g(x) = x + 1 \). Find \( f \circ g \) and \( g \circ f \).
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\[
(f \circ g)(x) = f(g(x)) = \sqrt{x + 1}
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Example

Define $f(x) = \sqrt{x}$ and $g(x) = x + 1$. Find $f \circ g$ and $g \circ f$.

\[
(f \circ g)(x) = f(g(x)) = \sqrt{x + 1}
\]

\[
(g \circ f)(x) = g(f(x)) = \sqrt{x} + 1
\]
Example

Define \( f(x) = \sqrt{x} \) and \( g(x) = x + 1 \). Find \( f \circ g \) and \( g \circ f \).

\[
(f \circ g)(x) = f(g(x)) = \sqrt{x + 1}
\]

\[
(g \circ f)(x) = g(f(x)) = \sqrt{x + 1}
\]

Note that \( f \circ g \neq g \circ f \).
Let \( h: F \to E \), \( g: E \to D \) and \( f: D \to C \), then

\[
(f \circ g) \circ h = f \circ (g \circ h).
\]
Let $h: F \to E$, $g: E \to D$ and $f: D \to C$, then

$$(f \circ g) \circ h = f \circ (g \circ h).$$

- Usually we omit the parenthesis: $f \circ g \circ h$. 

**Diagram:**

- $x \in F$
- $h(x) \in E$
- $g(h(x)) \in D$
- $f(g(h(x))) \in C$
Shifting in $y$-direction

The graph of $f(x) + c$ is obtained from the graph of $f$ by shifting it upward by $c$ units if $c > 0$, or downward by $|c|$ units if $c < 0$.

Example

$y = x^2$
$y = x^2 + 2$
$y = x^2 - 1$
Scaling in $y$-direction

- The graph of $cf(x)$ is obtained from the graph of $f$ by stretching it with a factor of $c$ units if $c > 1$, or shrinking it by $c$ units if $0 < c < 1$.
- If $c < 0$, then the graph is also reflected across the $x$-axis.

Example

- $y = \sqrt{x}$
- $y = 2\sqrt{x}$
- $y = \frac{1}{2}\sqrt{x}$
Horizontal shifting

Shifting in x-direction

The graph of $f(x + c)$ is obtained from the graph of $f$ by shifting it $c$ units to the left if $c > 0$, or to the right by $|c|$ units if $c < 0$.

Example

$y = x^2$

$y = (x + 2)^2$

$y = (x - 1)^2$

Functions - Horizontal Shift.nb
Horizontal scaling

Scaling in $x$-direction

- The graph of $f(cx)$ is obtained from the graph of $f$ by shrinking it with a factor of $c$ units if $c > 1$, or stretching it with a factor $c$ units if $0 < c < 1$.
- If $c < 0$, then the graph is also reflected across the $y$-axis.

Example

- $y = (x - 1)^2$
- $y = (2x - 1)^2$
- $y = \left(\frac{1}{2}x - 1\right)^2$
Reflecting in $x$ and $y$-direction

**Mirroring:** The graph of $f(-x)$ is obtained from the graph of $f$ by reflecting it across the $y$-axis.

**Flipping:** The graph of $-f(x)$ is obtained from the graph of $f$ by reflecting it across the $x$-axis.

---

**Example**

\[
\begin{align*}
  y &= \sqrt{x} \\
  y &= \sqrt{-x} \\
  y &= -\sqrt{x}
\end{align*}
\]
Assignment: IMM1 - Tutorial 1.2
The number $\pi$ is a fundamental constant in mathematics, representing the ratio of the circumference of a circle to its diameter. Mathematically, this is expressed as:

$$\pi = \frac{\text{circumference}}{\text{diameter}}$$

The value of $\pi$ is an irrational number, meaning it cannot be expressed exactly as a simple fraction, and its decimal representation goes on infinitely without repeating. The first few digits of $\pi$ are:

$$\pi \approx 3.141592653589793238462643383279502884197169399375105820\ldots$$
In a sector, the length of the arc is proportional to the angle of the sector and the radius of the circle.

\[ L \propto r \theta \quad \Rightarrow \quad L = k r \theta. \]
In a sector, the length of the arc is proportional to the angle of the sector and the radius of the circle.

- \( L \propto r \theta \implies L = k r \theta. \)
- The constant \( k \) depends on the units for measuring angles.
In a sector, the length of the arc is proportional to the angle of the sector and the radius of the circle.

- \( L \propto r \theta \quad \Rightarrow \quad L = k r \theta. \)
- The constant \( k \) depends on the units for measuring angles.
- The **radian** is a unit for angles such that \( k = 1 \).
Theorem

In a sector, the length of the arc is proportional to the angle of the sector and the radius of the circle.

- \( L \propto r \theta \Rightarrow L = k r \theta \).
- The constant \( k \) depends on the units for measuring angles.
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- A full circle is \( 2\pi \) radians.
Theorem

In a sector, the length of the arc is proportional to the angle of the sector and the radius of the circle.

\[ L \propto r \theta \quad \Rightarrow \quad L = k \, r \theta. \]

- The constant \( k \) depends on the units for measuring angles.
- The **radian** is a unit for angles such that \( k = 1 \).
- A full circle is \( 2\pi \) radians.
- If angles are measured in radians then \( L = r \theta \).
The Simpsons, (TM) Twentieth Century Fox
The Simpsons, (TM) Twentieth Century Fox

\[ \pi = \text{Pie} \]
Triangle $ABC$ is rectangular ($\angle ABC = \frac{\pi}{2}$), angle at $A$ is acute ($0 < \theta < \frac{\pi}{2}$).

$$\cos \theta = \frac{AB}{AC},$$

$$\sin \theta = \frac{BC}{AC},$$

$$\tan \theta = \frac{BC}{AB} = \frac{\frac{BC}{AC}}{\frac{AB}{AC}} = \frac{\sin \theta}{\cos \theta}.$$
Triangle $ABC$ is rectangular ($\angle ABC = \frac{\pi}{2}$), angle at $A$ is acute ($0 < \theta < \frac{\pi}{2}$).

\[
\cos \theta = \frac{AB}{AC}, \quad \sec \theta = \frac{1}{\cos \theta},
\]
\[
\sin \theta = \frac{BC}{AC}, \quad \csc \theta = \frac{1}{\sin \theta},
\]
\[
\tan \theta = \frac{BC}{AB} = \frac{BC}{AC} = \frac{\sin \theta}{\cos \theta},
\]
\[
\cot \theta = \frac{1}{\tan \theta}.
\]
For arbitrary angles, the sine, cosine are defined with the **unit circle**: the circle with center \((0, 0)\) and radius 1.

\[
\begin{align*}
\sin \theta & \quad \cos \theta \\
\end{align*}
\]
For arbitrary angles, the sine, cosine are defined with the **unit circle**: the circle with center \((0, 0)\) and radius 1.
Graphs of sine and cosine

\[ y = \sin \theta \]

\[ x = \cos \theta \]
Graphs of sine and cosine

\[y = \sin(\theta)\]

\[x = \cos(\theta)\]
Graph of the tangent

\[ \tan \theta \]
Sine, cosine and tangent of special angles

\[ \cos \left( \frac{\pi}{4} \right) = \frac{1}{2} \sqrt{2} \]
\[ \sin \left( \frac{\pi}{4} \right) = \frac{1}{2} \sqrt{2} \]
\[ \tan \left( \frac{\pi}{4} \right) = 1 \]

\[ \cos \left( \frac{\pi}{6} \right) = \frac{1}{2} \sqrt{3} \]
\[ \sin \left( \frac{\pi}{6} \right) = \frac{1}{2} \]
\[ \tan \left( \frac{\pi}{6} \right) = \frac{1}{3} \sqrt{3} \]
Sine and cosine are periodic:

\[ \cos x = \cos(x + 2\pi) \quad \text{for all } x \in \mathbb{R}, \]

\[ \sin x = \sin(x + 2\pi) \quad \text{for all } x \in \mathbb{R}. \]
Sine and cosine are periodic:

\[ \cos x = \cos(x + 2\pi) \quad \text{for all } x \in \mathbb{R}, \]
\[ \sin x = \sin(x + 2\pi) \quad \text{for all } x \in \mathbb{R}. \]

Sine and cosine are congruent:

\[ \cos x = \sin \left( x + \frac{1}{2}\pi \right) \quad \text{for all } x \in \mathbb{R}, \]
\[ \sin x = \cos \left( x - \frac{1}{2}\pi \right) \quad \text{for all } x \in \mathbb{R}. \]
**Symmetry**

- Sine and cosine are **symmetric**:

  \[
  \cos(-x) = \cos x \quad \text{for all } x \in \mathbb{R}, \text{ in other words: } \cos x \text{ is even},
  \]

  \[
  \sin(-x) = -\sin x \quad \text{for all } x \in \mathbb{R}, \text{ in other words: } \sin x \text{ is odd},
  \]
- Sine and cosine are **symmetric**:

  \[ \cos(-x) = \cos x \quad \text{for all } x \in \mathbb{R}, \text{ in other words: } \cos x \text{ is even}, \]

  \[ \sin(-x) = -\sin x \quad \text{for all } x \in \mathbb{R}, \text{ in other words: } \sin x \text{ is odd}, \]

- Sine and cosine are **half-periodic**:

  \[ \cos(x - \pi) = \cos(x + \pi) = -\cos x \quad \text{for all } x \in \mathbb{R}, \]
Sine and cosine are **symmetric**:

\[
\cos(-x) = \cos x \quad \text{for all } x \in \mathbb{R}, \text{ in other words: } \cos x \text{ is even},
\]

\[
\sin(-x) = -\sin x \quad \text{for all } x \in \mathbb{R}, \text{ in other words: } \sin x \text{ is odd},
\]

Sine and cosine are **half-periodic**:

\[
\cos(x - \pi) = \cos(x + \pi) = -\cos x \quad \text{for all } x \in \mathbb{R},
\]

\[
\sin(x - \pi) = \sin(x + \pi) = -\sin x \quad \text{for all } x \in \mathbb{R}.
\]
Assignment: IMM1 - Tutorial 1.3
Pythagoras’ theorem

\[
\cos^2 \theta + \sin^2 \theta = \left( \frac{c}{b} \right)^2 + \left( \frac{a}{b} \right)^2 = \frac{c^2}{b^2} + \frac{a^2}{b^2}
\]

\[
= \frac{c^2 + a^2}{b^2}
\]

\[
= \frac{b^2}{b^2} = 1.
\]
For any arbitrary triangle with angles $\alpha$, $\beta$, $\gamma$ and edge lengths $a$, $b$, $c$ as defined above, the following equations hold:

\[
\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.
\]
The law of sines

**Theorem**

For any arbitrary triangle with angles \( \alpha, \beta, \gamma \) and edge lengths \( a, b, c \) as defined above, the following equations hold:

\[
\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.
\]

**Proof:**

\[
\frac{\sin \alpha}{a} = \frac{h}{b} = \frac{h}{ab} = \frac{h}{a} = \frac{\sin \beta}{b}.
\]
**Theorem**

For arbitrary $\alpha, \beta \in \mathbb{R}$ we have

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$
Theorem

For arbitrary $\alpha, \beta \in \mathbb{R}$ we have

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$  

From the law of sines follows

$$\frac{\sin(\alpha + \beta)}{c} = \frac{\sin \varphi}{a},$$

hence

$$\sin(\alpha + \beta) = \frac{c \sin \varphi}{a}$$  

$$= \frac{c \ h/b}{a} = \frac{(c_1 + c_2) \ h}{a \ b}$$  

$$= \frac{c_1 \ h}{b \ a} + \frac{h \ c_2}{b \ a}$$  

$$= \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$
Theorem

For arbitrary $\alpha, \beta \in \mathbb{R}$ we have

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$
Theorem

For arbitrary $\alpha, \beta \in \mathbb{R}$ we have

$$
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.
$$
Theorem

For arbitrary $\alpha, \beta \in \mathbb{R}$ we have

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta,$$

and

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$ 

- We prove the first equation:
  $$\sin(\alpha - \beta) = \sin (\alpha + (-\beta))$$
  $$= \sin \alpha \cos (-\beta) + \cos \alpha \sin (-\beta)$$
  $$= \sin \alpha \cos \beta - \cos \alpha \sin \beta.$$
Example

Find an exact value for $\cos \frac{\pi}{12}$. 

Write $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$.

$\cos \left( \frac{\pi}{12} \right) = \cos \left( \frac{\pi}{3} - \frac{\pi}{4} \right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} = \frac{1}{2} \cdot \frac{1}{2} \sqrt{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \sqrt{2} = \frac{1}{2} \sqrt{2} + \frac{\sqrt{6}}{4}$.
Example

Find an exact value for $\cos \frac{\pi}{12}$.

Write $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$.
Example

*Find an exact value for* \( \cos \frac{\pi}{12} \).

- Write \( \frac{1}{12} = \frac{1}{3} - \frac{1}{4} \).

- \[
\cos \left( \frac{\pi}{12} \right) = \cos \left( \frac{\pi}{3} - \frac{\pi}{4} \right)
\]
Example

Find an exact value for $\cos \frac{\pi}{12}$.

- Write $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$.

- $\cos \left( \frac{\pi}{12} \right) = \cos \left( \frac{\pi}{3} - \frac{\pi}{4} \right)$
  
  $= \cos \left( \frac{\pi}{3} \right) \cos \left( \frac{\pi}{4} \right) + \sin \left( \frac{\pi}{3} \right) \sin \left( \frac{\pi}{4} \right)$
Example

Find an exact value for $\cos \frac{\pi}{12}$.

- Write $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$.

- \[
\cos \left( \frac{\pi}{12} \right) = \cos \left( \frac{\pi}{3} - \frac{\pi}{4} \right)
= \cos \left( \frac{\pi}{3} \right) \cos \left( \frac{\pi}{4} \right) + \sin \left( \frac{\pi}{3} \right) \sin \left( \frac{\pi}{4} \right)
= \frac{1}{2} \cdot \frac{1}{2} \sqrt{2} + \frac{1}{2} \sqrt{3} \cdot \frac{1}{2} \sqrt{2}
= \frac{\sqrt{2} + \sqrt{6}}{4}
\]
Theorem

For arbitrary $\alpha \in \mathbb{R}$ we have

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha,$$

and

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha.$$

We prove the first equation:

$$\sin(2\alpha) = \sin(\alpha + \alpha)$$

$$\quad = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$$

$$\quad = 2 \sin \alpha \cos \alpha.$$
Theorem

For arbitrary $\alpha \in \mathbb{R}$ we have

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha,$$

and

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha.$$  

- We prove the first equation:

  $$\sin(2\alpha) = \sin(\alpha + \alpha)$$
  $$= \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$$
  $$= 2 \sin \alpha \cos \alpha.$$
**Theorem**

For arbitrary $\alpha \in \mathbb{R}$ we have

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha,$$

and

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha.$$

We prove the first equation:

$$\sin(2\alpha) = \sin(\alpha + \alpha)$$

$$= \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$$

$$= 2 \sin \alpha \cos \alpha.$$
Theorem

For arbitrary $\alpha \in \mathbb{R}$ we have

$$\sin^2 \left(\frac{1}{2} \alpha \right) = \frac{1 - \cos \alpha}{2},$$

and

$$\cos^2 \left(\frac{1}{2} \alpha \right) = \frac{1 + \cos \alpha}{2}.$$

We prove the first equation. Let $\varphi = \frac{1}{2} \alpha$, then

$$\frac{1 - \cos \alpha}{2} = \frac{1 - \cos(2\varphi)}{2} = \frac{1 - (\cos^2 \varphi - \sin^2 \varphi)}{2} = \frac{1 - \cos^2 \varphi + \sin^2 \varphi}{2} = \frac{\sin^2 \varphi}{2} + \frac{\sin^2 \varphi}{2} = \sin^2 \varphi = \sin^2 \left(\frac{1}{2} \alpha \right).$$
Example

Find an exact value for \( \cos \frac{\pi}{12} \).

Use the half-formula for cosine:

\[
\cos 2\left(\frac{\pi}{12}\right) = \cos \left(\frac{1}{2} \cdot \frac{\pi}{6}\right) = 1 + \cos \frac{\pi}{6}^2 = 1 + \frac{1}{2} \sqrt{3} = \frac{1}{2} + \frac{1}{4} \sqrt{3}.
\]

Hence \( \cos \frac{\pi}{12} = \pm \sqrt{\frac{1}{2} + \frac{1}{4} \sqrt{3}} \).

Since \( \frac{\pi}{12} \) is an acute angle, the cosine of \( \frac{\pi}{12} \) must be positive, so \( \cos \frac{\pi}{12} = \sqrt{\frac{1}{2} + \frac{1}{4} \sqrt{3}} \).
Example

Find an exact value for $\cos \frac{\pi}{12}$.

- Use the half-formula for cosine:

$$\cos^2 \left( \frac{\pi}{12} \right) = \cos^2 \left( \frac{1}{2} \cdot \frac{\pi}{6} \right) = \frac{1 + \cos \frac{\pi}{6}}{2} = \frac{1 + \frac{1}{2}\sqrt{3}}{2} = \frac{1}{2} + \frac{1}{4}\sqrt{3}.$$
Example

Find an exact value for $\cos \frac{\pi}{12}$.

- Use the half-formula for cosine:
  \[ \cos^2 \left( \frac{\pi}{12} \right) = \cos^2 \left( \frac{1}{2} \cdot \frac{\pi}{6} \right) = \frac{1 + \cos \frac{\pi}{6}}{2} = \frac{1 + \frac{1}{2}\sqrt{3}}{2} = \frac{1}{2} + \frac{1}{4}\sqrt{3}. \]

- Hence $\cos \frac{\pi}{12} = \pm \sqrt{\frac{1}{2} + \frac{1}{4}\sqrt{3}}$. 
Example

Find an exact value for $\cos \frac{\pi}{12}$.

- Use the half-formula for cosine:

$$\cos^2 \left( \frac{\pi}{12} \right) = \cos^2 \left( \frac{1}{2} \cdot \frac{\pi}{6} \right) = \frac{1 + \cos \frac{\pi}{6}}{2} = \frac{1 + \frac{1}{2}\sqrt{3}}{2} = \frac{1}{2} + \frac{1}{4}\sqrt{3}.$$  

- Hence $\cos \frac{\pi}{12} = \pm \sqrt{\frac{1}{2} + \frac{1}{4}\sqrt{3}}$.

- Since $\frac{\pi}{12}$ is an acute angle, the cosine of $\frac{\pi}{12}$ must be positive, so

$$\cos \frac{\pi}{12} = \sqrt{\frac{1}{2} + \frac{1}{4}\sqrt{3}}$$
- On previous slide: \( \cos \frac{\pi}{12} = \sqrt{\frac{1}{2}} + \frac{1}{4} \sqrt{3} \),

on slide 40 : \( \cos \frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4} \),
Wait a minute...

- On previous slide: \( \cos \frac{\pi}{12} = \sqrt{\frac{1}{2} + \frac{1}{4}\sqrt{3}}, \)

  on slide 40: \( \cos \frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4}, \)

- Square both results:

\[
\left( \sqrt{\frac{1}{2} + \frac{1}{4}\sqrt{3}} \right)^2 = \frac{1}{2} + \frac{1}{4}\sqrt{3},
\]

\[
\left( \frac{\sqrt{2} + \sqrt{6}}{4} \right)^2 = \frac{2 + 6 + 2\sqrt{2}\sqrt{6}}{16} = \frac{8 + 4\sqrt{3}}{16} = \frac{1}{2} + \frac{1}{4}\sqrt{3}.
\]
### Overview

| Periodicity | \( \sin(\alpha + 2\pi) = \sin \alpha \) and \( \sin(\alpha + \pi) = -\sin \alpha \)  
|             | \( \cos(\alpha + 2\pi) = \cos \alpha \) and \( \cos(\alpha + \pi) = -\cos \alpha \) |
| Symmetry    | \( \sin(-\alpha) = -\sin \alpha \)  
|             | \( \cos(-\alpha) = \cos \alpha \) |
| Congruence  | \( \sin \left( \alpha + \frac{\pi}{2} \right) = \cos \alpha \) and \( \sin \left( \alpha - \frac{\pi}{2} \right) = -\cos \alpha \)  
|             | \( \cos \left( \alpha + \frac{\pi}{2} \right) = -\sin \alpha \) and \( \cos \left( \alpha - \frac{\pi}{2} \right) = \sin \alpha \) |
| Sum formulas | \( \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \)  
|             | \( \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \) |
| Difference formulas | \( \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \)  
|              | \( \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \) |
| Doubling formulas | \( \sin(2\alpha) = 2 \sin \alpha \cos \alpha \)  
|              | \( \sin^2 \left( \frac{1}{2} \alpha \right) = \frac{1}{2} - \frac{1}{2} \cos \alpha \)  
|              | \( \cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha \)  
|              | \( \cos^2 \left( \frac{1}{2} \alpha \right) = \frac{1}{2} + \frac{1}{2} \cos \alpha \) |
| Pythagoras’ thm | \( \cos^2 \alpha + \sin^2 \alpha = 1 \) |
Assignment: IMM1 - Tutorial 1.4